

MAZIMUM-SIZED MATROIDS WITH NO MINORS

ISOMORPHIC TO  $U_{2,5}$ ,  $F_7$ ,  $F_7^-$ , OR  $P_7$

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Let  $\mathcal{M}$  be the class of simple matroids which do not contain the 5-point line  $U_{2,5}$ , the Fano plane  $F_7$ , the non-Fano plane  $F_7^-$ , or the matroid  $P_7$  as minors. Let  $h(n)$  be the maximum number of points in a rank- $n$  matroid in  $\mathcal{M}$ . We show that  $h(2) = 4$ ,  $h(3) = 7$ , and  $h(n) = n(n+1)/2$  for  $n > 3$ , and we also find all the maximum-sized matroids for each rank.

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## CHAPTER 1

### Preliminaries

The term *matroid* was first used by Whitney [WHNY] in 1935. There are several equivalent ways to define a matroid as shown by Oxley [OX1]. The first one we will look at is the definition of a matroid using the independent sets.

A *matroid*  $M$  is an ordered pair  $(E, \mathcal{I})$  consisting of a finite set  $E$  and a collection  $\mathcal{I}$  of subsets of  $E$  satisfying the following three conditions:

- (I1)  $\emptyset \in \mathcal{I}$ .
- (I2) If  $I \in \mathcal{I}$  and  $J \subseteq I$ , then  $J \in \mathcal{I}$ .
- (I3) If  $I_1$  and  $I_2$  are in  $\mathcal{I}$  and  $|I_1| < |I_2|$ , then there is an element  $e$  of  $I_2 - I_1$  such that  $I_1 \cup \{e\} \in \mathcal{I}$ .

Condition (I3) is called the *independence augmentation axiom*. The members of  $\mathcal{I}$  are the *independent sets* of  $M$ , and  $E$  is the *ground set* of  $M$ .

For an example of a matroid, let  $V$  be a vector space and let  $E$  be a finite set of vectors from  $V$ . Let  $\mathcal{I}$  be the set of linearly independent subsets of vectors from  $E$ . Then  $M = (E, \mathcal{I})$  is a matroid. The first axiom says that the set containing no vectors is linearly independent, the second axiom says that a subset of a set of linearly independent vectors is linearly independent, and the third axiom states that the subspace generated by a set of linearly independent vectors cannot be contained in the subspace of a proper subset of these vectors.

Uniform matroids give us another example of a type of matroid. Let  $r \leq n$  be two

nonnegative integers, and let  $E$  be an  $n$ -element set. Then let  $\mathcal{I}$  be the collection of all subsets of  $E$  with at most  $r$  elements. Then  $M = (E, \mathcal{I})$  is a matroid. It is called the *uniform matroid* of rank  $r$  on  $n$  elements and it is denoted by  $U_{r,n}$ .

Given a matroid  $M = (E, \mathcal{I})$ , we call a *basis* a maximal independent set. Subset of  $E$  which are not in  $\mathcal{I}$  are called the *dependent sets*. The minimal dependent sets are called *circuits*. From circuits, we can give another axiom system for matroids.

A *matroid*  $M$  is an ordered pair  $(E, \mathcal{C})$  consisting of a finite set  $E$  and a collection  $\mathcal{C}$  of subsets of  $E$  satisfying the following three conditions:

(C1)  $\emptyset \notin \mathcal{C}$ .

(C2) If  $C_1$  and  $C_2$  are members of  $\mathcal{C}$  and  $C_1 \subseteq C_2$ , then  $C_1 = C_2$ .

(C3) If  $C_1$  and  $C_2$  are distinct members of  $\mathcal{C}$  and  $e \in C_1 \cap C_2$ , then there is a member  $C_3$  of  $\mathcal{C}$  such that  $C_3 \subseteq (C_1 \cup C_2) - e$ .

Condition (C3) is called the *circuit elimination axiom*, and all the members of  $\mathcal{C}$  are the circuits of  $M$ . It is shown in [OX1] that the circuit axioms and the independent-set axioms are equivalent.

As defined in [OX1], a *graph*  $G$  consists of a non-empty set  $V(G)$  of *vertices* and a multiset  $E(G)$  of *edges* each of which consists of an unordered pair of vertices. A *walk*  $W$  in a graph  $G$  is a sequence  $v_0 e_1 v_1 e_2 \dots v_{k-1} e_k v_k$  such that  $v_0, v_1, \dots, v_k$  are vertices and  $e_1, e_2, \dots, e_k$  are edges, and each vertex or edge in the sequence, except  $v_k$ , is incident with its successor in the sequence. If the vertices  $v_0, v_1, \dots, v_k$  are distinct then the edges  $e_1, e_2, \dots, e_k$  are also distinct and  $W$  is a *path*. The *end-vertices* of this path are  $v_o$  and  $v_k$ , and the path is said to be a  $(v_o, v_k)$ -*path*. If  $P$  is a  $(u, v)$ -path in a graph  $G$  and  $e$  is an edge of  $G$  that joins  $u$  to  $v$  but is not in  $P$ , then the subgraph of

$G$  whose vertex set is  $V(P)$  and whose edge set is  $E(P) \cup \{e\}$  is called a *cycle*. Oxley [OX1] showed that given a graph  $G$  and the set  $E$  of all edges of  $G$ , and given  $\mathcal{C}$  as the set of cycles of  $G$ , then  $\mathcal{C}$  is the set of circuits in a matroid  $E$ . The *complete graph*,  $K_n$ , is the graph containing  $n$  vertices in which any pair of vertices is connected by an edge. We will be using  $M(K_n)$  throughout this paper to denote the cycle matroid of a  $K_n$ .

Given a matroid  $M = (E, \mathcal{I})$ , the *rank* of a subset  $N$  of  $E$  is the cardinality of the largest independent set contained in  $N$ . An extremely useful tool in matroid theory is the *geometric representation* of a matroid. An  $n$ -element independent set  $I$  is represented by  $n$  distinct points in  $(n - 1)$ -dimensional space and has rank  $n$ . One-element circuits or *loops* are marked in a single inset and have rank 0. Given a matroid  $M = (E, \mathcal{C})$ , a maximal subset  $X$  of  $E$  is said to be a *parallel class* if any two elements in  $X$  form a circuit. In a geometric representation, a parallel class is represented by a multiple point and has rank 1. A circuit which contains three elements is represented by three distinct collinear points and has rank 2. A circuit containing four elements is represented by four distinct points in a plane, no three of which are collinear, and has rank 3. In general, a circuit  $C$  containing  $n$  elements has rank  $= n - 1$  and is represented by  $n$  distinct points in  $(n - 2)$ -dimensional space, while any subset  $I$  of  $C$  containing  $k$  elements where  $k < n$  would be in  $(k - 1)$ -dimensional space.

A *simple matroid* is a matroid which contains no loops or parallel classes. Throughout this paper we will only be concerned with simple matroids. Given a geometric representation of a matroid  $M$  and a point  $a \in M$ , we can perform two different kinds

of operations on  $M$  by  $a$  called deletion by a point and contraction by a point. We can delete  $a$  simply by removing it. The independent sets in  $M \setminus a$  are the same as the independent sets in  $M$  which do not contain  $a$ . We call the new matroid  $M \setminus a$ , and this matroid is called the *deletion* of  $M$  by  $a$ . Contracting by  $a$  is a bit more complicated. We use  $M/a$  to denote the *contraction* of  $M$  by  $a$ , and the  $M/a$ -independent sets are the subsets  $I$  of  $S \setminus a$  such that  $I \cup \{a\}$  is independent in  $M$ . If  $M$  has rank- $n$ , the easiest way to visualize contraction by the point  $a$  is to fix any rank- $(n-1)$  subspace  $N$  of  $M$  which does not contain  $a$  as the “screen”. Then think of  $a$  “projecting” any points outside this subspace onto the “screen”. Finally,  $a$  itself is removed in the process of contraction. In essence, what happens is that lines containing  $a$  in  $M$  become points or multiple points in  $M/a$ . For the purposes of this dissertation, a *point* is a closed rank-1 matroid, so when counting points in a matroid, a multiple point will be counted as one point. A *minor*  $N$  of a matroid  $M$  is a matroid obtained by a sequence of contractions and deletions. Minors will be used extensively throughout this dissertation.



## CHAPTER 2

### Introduction

The starting point of this paper is a classical theorem by Heller [HE] which states: *Let  $M$  be a binary simple matroid of rank  $n$  with no  $F_7$ -minor. Then the number of points in  $M$  is at most  $\binom{n+1}{2}$ .* Kung and Oxley [K1], [KOX], Oxley, Vertigan, and Whittle [OVW], and Semple [SE], proved analogues of Heller's theorem. In this paper, we prove another analogue of Heller's theorem.

The following matroids occur throughout this paper. The matroid  $U_{2,q+2}$  is the  $(q+2)$ -point line. The matroid  $F_7$  is the Fano plane, and the matroid  $F_7^-$  is the non-Fano plane. The matroid  $M(K_n)$  is the cycle matroid of the complete graph on  $n$  vertices. The matroid  $P_7$  is the rank-3 matroid shown in Figure 2.1.

The matroid  $P$  is the parallel connection of two 4-point lines at a point, and  $O_1$  and  $O_2$  are the two non-isomorphic rank-4 parallel connections of three 4-point lines as shown in Figure 2.2.

The *Hopper matroid*  $H$  is the rank-3 matroid shown in Figure 2.3.

The matroids  $P_1(H, H)$  and  $P_2(H, H)$  are the two non-isomorphic rank-4 parallel connections of two  $H$ 's at a 4-point line, and the matroid  $P(H, K_4)$  is the parallel connection of an  $H$  and an  $M(K_4)$  at a 3-point line as shown in Figure 2.4.

Finally, the matroids  $R_1$ ,  $R_2$ , and  $R_3$  are the three non-isomorphic rank-4 parallel connections of an  $H$  and a  $U_{2,4}$  at a point as shown in Figure 2.5.

A *point* is a rank-1 flat. For a class  $\mathcal{M}$  of matroids, let  $h(n)$  be the maximum

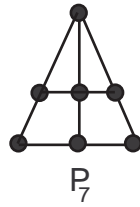


Figure 2.1: The matroid  $P_7$

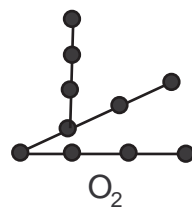
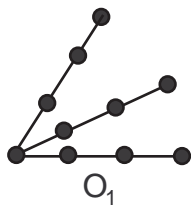


Figure 2.2: The matroids  $O_1$  and  $O_2$

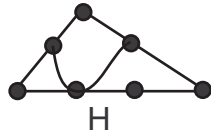


Figure 2.3: The Hopper Matroid

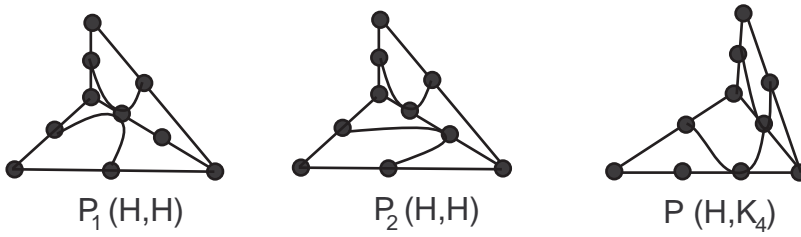


Figure 2.4: The matroids  $P_1(H, H)$ ,  $P_2(H, H)$ , and  $P(H, K_4)$

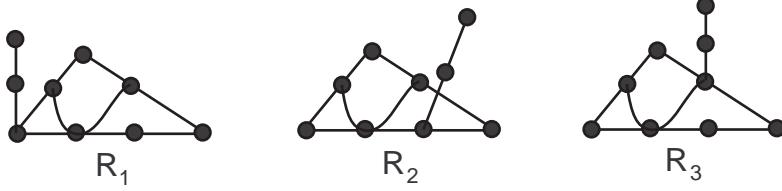


Figure 2.5: The matroids  $R_1$ ,  $R_2$ , and  $R_3$

number of points in a rank- $n$  matroid in  $\mathcal{M}$ . Then  $h(n)$  is called the *size function* of  $\mathcal{M}$ . We prove the following theorem.

**Theorem 2.1** *Let  $\mathcal{M}$  be the class of all simple matroids which do not contain  $U_{2,5}$ ,  $F_7$ ,  $F_7^-$ , or  $P_7$  as a minor. Then  $\mathcal{M}$  has the size function  $h(2) = 4$ ,  $h(3) = 7$ , and  $h(n) = \binom{n+1}{2}$  for  $n \geq 4$ . The maximum-sized matroids in  $\mathcal{M}$  are as follows:*

rank	maximum-sized matroid
2	$U_{2,4}$
3	$P, H$
4	$O_1, O_2, R_1, R_2, R_3, P_1(H, H),$ $P_2(H, H), P(H, K_4), M(K_5)$
$n \geq 5$	$M(K_{n+1})$

## CHAPTER 3

### Technical Lemmas

Throughout the main proof we will be using a couple of technical lemmas quite frequently. The most important of these is the Long-Line Lemma. A long line is any line which contains three or more points.

**Lemma 3.1 Long-Line Lemma:** *Let  $\mathcal{M}$  be the class of all simple matroids which do not contain  $U_{2,5}$ ,  $F_7$ ,  $F_7^-$ , or  $P_7$  as a minor. Let  $M$  be a rank- $n$  matroid in  $\mathcal{M}$ . Then  $M$  does not contain a point  $a$  which is on  $n$  long lines.*

**Proof.** Clearly, if  $M$  is rank-1 then there are no long lines, and if  $M$  is rank-2 then there is only one long line, so the lemma holds.

Let  $M$  be a rank-3 matroid in  $\mathcal{M}$ , and let  $a$  be a point in  $M$  which is on three long lines. We can assume without loss of generality that  $M$  contains exactly seven points. Because  $M$  has no  $U_{2,5}$ , then each point in  $M$  must be on at least two long lines. But then by considering all the possibilities, it is easy to see that  $M$  must be a  $P_7$ ,  $F_7$ , or an  $F_7^-$ , so no point in  $M$  can be on three long lines.

To finish the proof we will proceed by induction. Assume that  $M$  is rank- $n$  matroid in  $\mathcal{M}$ , where  $n \geq 4$ , and that there exists a point  $a$  in  $M$  which is on  $n$  long lines,  $l_1, l_2, \dots, l_n$ . Note that no three of these lines can be coplanar, or else this would contradict the rank-3 case. Assume that if  $N$  is any rank- $k$  matroid in  $\mathcal{M}$ , where  $k < n$ , then  $N$  has no points which are on  $k$  long lines. Let  $b$  be a point on  $l_1$  besides  $a$ . Because no point is on three coplanar long lines, this implies that  $a$  is on  $n - 1$

long lines in  $M/b$ , a rank- $l$  space where  $l \leq (n - 1)$ . But this is a contradiction, so the lemma holds for all ranks. ■

**Corollary 3.2** *Let  $\mathcal{M}$  be the class of all simple matroids which do not contain  $U_{2,5}$ ,  $F_7$ ,  $F_7^-$ , or  $P_7$  as a minor. Let  $M$  be a rank- $n$  matroid in  $\mathcal{M}$ , and assume that  $M$  contains a point  $a$  which is on  $n - 1$  long lines. Then all points in  $M$  are on a long line containing  $a$ , or on a plane spanned by two long lines intersecting at the point  $a$ .*

**Proof.** Let  $M$  be a rank- $n$  matroid in  $\mathcal{M}$ , and let  $a$  be a point in  $M$  which is on  $n - 1$  long lines. Let  $b$  be a point in  $M$  which is neither on a long line containing  $a$  nor on any plane spanned by two long lines intersecting at the point  $a$ . Then  $M/b$  is a rank- $(n - 1)$  space containing a point which is on  $n - 1$  long lines, a contradiction of Lemma 3.1. ■

**Lemma 3.3 Connectivity Lemma:** *Let  $\mathcal{M}$  be a class of matroids with size function  $h(m) = ((m - 1)q) + 1$  for  $q \geq 2$  and  $m \leq (2q - 2)$ , and  $h(m) = \binom{m+1}{2}$  for  $m \geq (2q - 1)$ . Let  $M$  be a rank- $n$  matroid in  $\mathcal{M}$  where  $n \geq (2q - 1)$  and assume that  $|M| = \binom{n+1}{2}$ . Then  $M/a$  is connected for all points  $a$  in  $M$ .*

**Proof.** Let  $a$  be a point in  $M$  and assume for a contradiction that  $M/a$  is disconnected. Then  $M/a$  can be divided into two disjoint components,  $N_1$  and  $N_2$ . Let  $n_1$  be the rank of  $N_1$ , and  $n_2$  be the rank of  $N_2$ . Then  $n = n_1 + n_2 + 1$ . Let  $P_1$  be the submatroid of  $M$  such that  $P_1/a = N_1$ , and let  $P_2$  be the submatroid of  $M$  such the  $P_2/a = N_2$ . Then the rank of  $P_1$  is  $n_1 + 1$  and the rank of  $P_2$  is  $n_2 + 1$ . Also,

we have that  $|M| = |P_1| + |P_2| - 1$ , since  $P_1$  and  $P_2$  are disjoint in  $M$  except for  $a$ , and all points in  $M$  are in  $P_1$  or  $P_2$ . We will show that this last statement is false by considering cases based on the ranks of  $P_1$  and  $P_2$ .

**Case 1:** Assume that  $p_1 \leq (2q - 2)$  and  $p_2 \leq (2q - 2)$ . Since  $|M| = |P_1| + |P_2| - 1$ , then we get  $\binom{n+1}{2} = [q(p_1 - 1) + 1] + [q(p_2 - 1) + 1] - 1$ . Expanding and cancelling terms, we obtain  $(p_1 + p_2)n + (4q - 2) = (p_1 + p_2)2q$ , which is impossible if  $q \geq 2$  and  $n \geq 2q - 1$ .

**Case 2:** Assume without loss of generality that  $p_1 > (2q - 2)$  and  $p_2 \leq (2q - 2)$ . Again using  $|M| = |P_1| + |P_2| - 1$ , we get  $\binom{n+1}{2} = \binom{p_1+1}{2} + [q(p_2 - 1) + 1] - 1$ . Using the fact that  $\binom{m}{2} = 1 + 2 + 3 + \dots + (m - 1)$ , we get  $(p_1 + 1) + \dots + (p_1 + p_2 - 1) = q(p_2 - 1)$ . But since  $p_1 \geq 2q - 1$ , then  $(p_1 + 1) + \dots + (p_1 + p_2 - 1) > 2q(p_2 - 1) > q(p_2 - 1)$ , a contradiction.

**Case 3:** Assume that  $p_1 > 2q - 2$  and  $p_2 > 2q - 2$ . Then using  $|M| = |P_1| + |P_2| - 1$ , we get  $\binom{n+1}{2} = \binom{p_1+1}{2} + \binom{p_2+1}{2} - 1$ . Expanding and cancelling terms, we obtain  $p_1 p_2 = p_1 + p_2 - 1$ , which is impossible if  $p_1 \geq 2$  and  $p_2 \geq 2$ .

Therefore our assumption is false, so  $M/a$  must be connected. ■

**Lemma 3.4** *Let  $M$  be a simple rank- $n$  matroid on  $\binom{n+1}{2}$  points. Suppose the points can be labelled  $e_i, 1 \leq i \leq n, e_{ij}, 1 \leq i < j \leq n$  so that  $\{e_1, \dots, e_n\}$  is a basis,  $e_i, e_{ij}, e_j$  are collinear, and  $e_{ij}, e_{jk}, e_{ik}$  are collinear. Then  $M$  is an  $M(K_{n+1})$ .*

The proof consists of showing that all the circuits of  $M(K_{n+1})$  can be derived by circuit elimination from the 3-element circuits. This is done in p. 493 of [KK]. In particular, the matroid structure of  $M(K_{n+1})$  is determined by its 3-element circuits

and any other matroid with the same 3-element circuits must be isomorphic to it.

Now we are ready to prove the theorem. Clearly the maximum-sized rank-1 matroid is a point and the maximum-sized rank-2 matroid is a 4-point line. We first consider the rank-3, rank-4, and rank-5 cases and then we finish the proof of the theorem by using induction for ranks  $\geq 6$ . For each rank, we first show that any maximum-sized matroid must be on our list, and then we show that if a matroid is larger than the maximum size for that rank, then it must contain one of the forbidden minors.



## CHAPTER 4

### Rank-3 Case

Let  $M$  be a rank-3 matroid in  $\mathcal{M}$  and assume that  $|M| = 7$ . We want to show that  $M$  is isomorphic to a  $P$  or an  $H$ .

**Case 1:** Assume that  $M$  contains at least one 4-point line,  $l$ . If  $\text{rank}(M \setminus l) = 2$ , then  $M$  must be a  $P$ . If  $\text{rank}(M \setminus l) = 3$ , then each of the points in  $M$  which are not on  $l$  must be on two 3-point lines, or else  $M$  contains  $U_{2,5}$  as a minor. Thus  $M$  is an  $H$ . (also see Figure 2.3)

**Case 2:** Assume that  $M$  contains no 4-point lines. Then every point  $a$  in  $M$  is on at least two 3-point lines, or else  $|M/a| \geq 5$  which implies that  $M$  contains a  $U_{2,5}$  as a minor. Also no point can be on three long lines by Lemma 3.1, so every point is on exactly two 3-point lines.

We finish this case with a simple counting argument to show that this case cannot happen. Let  $S = \{(a, L) | a \text{ is a point in } M, \text{ and } L \text{ is a 3-point line in } M \text{ which contains } a\}$ . Since there are seven points and each point is on exactly two 3-point lines, then  $|S| = 14$ . But each line is counted exactly three times, so  $|S|$  is divisible by 3, a contradiction. This implies that  $M$  must contain at least one 4-point line.

We have now shown that if  $|M| = 7$  and  $M$  is in  $\mathcal{M}$ , then  $M$  is a  $P$  or an  $H$ . To finish this case it suffices to show that any rank-3 one-point extension of a  $P$  or an  $H$  contains a forbidden minor. But the only 1-point extensions of a  $P$  or an  $H$  which do not contain a  $U_{2,5}$  as a minor are shown in Figure 4.1, and these contain a forbidden



Figure 4.1:

minor by Lemma 3.1.

## CHAPTER 5

### Rank-4 Case

Let  $M$  be a rank-4 matroid in  $\mathcal{M}$  and assume that  $|M| = 10$ . We now consider cases based on the number of 4-point lines that intersect at a single point. Our goal is to show that in all cases,  $M$  either contains one of the forbidden minors, or that  $M$  is isomorphic to  $O_1$ ,  $O_2$ ,  $R_1$ ,  $R_2$ ,  $R_3$ ,  $P(H, K_4)$ ,  $P_1(H, H)$ ,  $P_2(H, H)$ , or  $M(K_5)$ .

**Case 1:** Assume that there exists a point  $a$  in  $M$  which is incident on at least two 4-point lines,  $l_1$  and  $l_2$ . Let  $b$ ,  $c$ , and  $d$ , represent the three extra points which are not on  $l_1$  or  $l_2$ . Since any 8-point plane contains a forbidden minor, none of the extra points can be in the plane  $l_1 \vee l_2$ , and any two of  $b$ ,  $c$ , or  $d$ , must be on a long line with one of the points on  $l_1$  or  $l_2$ . This implies that  $b$ ,  $c$ , and  $d$ , must all be in the plane  $l_1 \vee b$  or the plane  $l_2 \vee b$ , meaning that one of these planes contains seven points and must be a  $P$  or an  $H$ . If one of the planes is a  $P$ , then  $M$  is an  $O_1$  or an  $O_2$ , while if one of the planes is an  $H$ , then  $M$  is an  $R_1$  or an  $R_2$ .

**Case 2:** Assume that there exist a point  $a$  in  $M$  which is incident on one 4-point line,  $l_1$ , and one 3-point line,  $l_2$ . Let  $b$ ,  $c$ ,  $d$ , and  $e$  be the four points in  $M$  not on  $l_1$  or  $l_2$ , at least three of which must be outside the plane  $l_1 \vee l_2$ . The first two subcases we consider are when contraction by one of these extra points outside the plane  $l_1 \vee l_2$ , say  $b$ , produces a  $P$  or an  $H$ . The last subcase we consider is  $M/b$ ,  $M/c$ ,  $M/d$ , and  $M/e$  are all 6-point planes.

**Case 2a:** Suppose that  $M/b$  is a  $P$ . Then  $c$ ,  $d$ , and  $e$  must all be in the plane

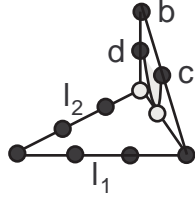


Figure 5.1:

$l_1 \vee b$  or the plane  $l_2 \vee b$ . We want to show that all of the extra points must be in the plane  $l_2 \vee b$ . Assume not. Without loss of generality, let  $c$  be in the plane  $l_1 \vee b$  but not in the plane  $l_2 \vee b$ . At least one point, say  $d$ , must be in the plane  $l_2 \vee b$  but not the plane  $l_1 \vee b$ , since  $b$  must contract at least one point to  $l_2$  which becomes a 4-point line in  $M/b$ . But then  $M/d$  contains at least eight points as shown in Figure 5.1, a contradiction.

Now all four extra points are in the plane  $l_2 \vee b$ , and since  $l_2$  is in this plane and  $l_2$  is a 3-point line, then this plane must be an  $H$ . This implies that  $M$  is an  $R_1$  or an  $R_3$ .

**Case 2b:** Suppose that  $M/b$  is an  $H$ . Without loss of generality, let  $c$  be a point which becomes the point in  $M/b$  which is not on  $l_1$  or  $l_2$ . In  $M/b$ ,  $c$  is on two 3-point lines. Let  $f$  be a point on  $l_1$ , and let  $g$  be a point on  $l_2$  such that  $f$ ,  $g$ , and  $c$  are on a long line in  $M/b$  as shown in Figure 5.2.

The first thing we show is that at least one of  $c$ ,  $d$ , or  $e$  is in the plane  $l_1 \vee l_2$ .

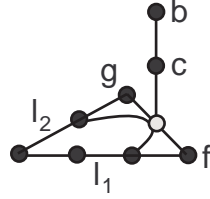


Figure 5.2:

Assume not. We know that  $c$  is in the plane  $b \vee f \vee g$ , by definition. We know that  $d$  and  $e$  must be in the planes  $l_1 \vee b$ ,  $l_2 \vee b$ , or  $b \vee f \vee g$ , since otherwise  $M/b$  would contain at least eight points. But this fact implies that  $d$  and  $e$  must both be in the plane  $b \vee f \vee g$  or else  $M/c$  would contain at least eight points. So this means that  $b$ ,  $c$ ,  $d$ , and  $e$  are all in the plane  $b \vee f \vee g$ , but none of these points are in the plane  $l_1 \vee l_2$ . Let  $h$  be the third point on  $l_2$ , besides  $g$  and  $a$ . But then by referring back to Figure 5.2, it is easy to see that  $M/h$  is an 8-point plane, a contradiction.

We have now shown that the plane  $l_1 \vee l_2$  contains seven points, and since it contains only one 4-point line it must be an  $H$ , and so one of the extra points, let's say  $e$ , must be in this plane. Then any two of  $b$ ,  $c$ , and  $d$ , must be collinear with one of the points in the plane  $l_1 \vee l_2$ , or else  $M$  contains an 8-point plane as a minor. This also implies that  $b$ ,  $c$ , and  $d$ , must be coplanar with one of the long lines in the plane  $l_1 \vee l_2$ . If  $b$ ,  $c$ , and  $d$ , are on a 4-point line with one of the points in the plane  $l_1 \vee l_2$ , then  $M$  is an  $R_1$ ,  $R_2$ , or  $R_3$ . If  $b$ ,  $c$ , and  $d$ , are not on a 4-point line, but are coplanar

with a 3-point line in the plane  $l_1 \vee l_2$ , then  $M$  is a  $P(H, K_4)$ . Finally, if  $b$ ,  $c$ , and  $d$ , are not on a 4-point line, but are coplanar with a 3-point line in the plane  $l_1 \vee l_2$ , then  $M$  is a  $P_1(H, H)$  or a  $P_2(H, H)$ .

**Case 2c:** The last case to consider is when contraction by  $b$ ,  $c$ ,  $d$ , and  $e$ , all produce a 6-point plane. This implies that none of these points can be in the plane  $l_1 \vee l_2$ . By using a similar argument as that used in case 2a, we can show that  $b$ ,  $c$ ,  $d$ , and  $e$ , must all be in either the plane  $l_1 \vee b$  or the plane  $l_2 \vee b$ . But any 8-point plane contains a forbidden minor, so  $b$ ,  $c$ ,  $d$ , and  $e$ , must all be in the plane  $l_2 \vee b$ . Therefore the plane  $l_2 \vee b$  contains seven points and a three-point line,  $l_2$ , so it must be an  $H$ , implying that  $M$  is an  $R_1$  or an  $R_3$ .

**Case 3:** Let  $l$  be a 4-point line in  $M$  such that no point on  $l$  is on any other long lines. Then  $M$  can have at most 4 different planes containing  $l$ , or else contraction by any point on  $l$  produces a  $U_{2,5}$ . But each plane containing  $l$  can contain at most one extra point besides the points on  $l$ , or else contraction by one of the extra points would produce a  $U_{2,5}$ . This implies that  $M$  can contain at most 8-points, so that implies that this case cannot happen. Therefore if  $M$  contains a 4-point line, at least one of the points on this line must be on another long line.

**Case 4:** For the last case, we assume the  $M$  contains no 4-point lines. This implies that every point  $a$  in  $M$  must be on at least two 3-point lines, or else  $|M/a| \geq 8$ , which would be a contradiction by the rank-3 case. Also, no point in  $M$  can be on four or more 3-point lines by the Lemma 3.1.

Let's assume for a contradiction that no point in  $M$  is on three 3-point lines. Then every point is on exactly two 3-point lines. Let  $S = \{(a, L) | a \text{ is a point in } M, \text{ and}$

$L$  is a three point line in  $M$  which contains  $a$ }. Since there are ten points and each point is on exactly two 3-point lines, then  $|S| = 20$ . But each line is counted exactly three times, so  $|S|$  is divisible by three, a contradiction, so  $M$  contains at least one point which is on three 3-point lines.

Let  $a$  be a point in  $M$  which is on exactly three 3-point lines,  $l_1$ ,  $l_2$ , and  $l_3$ , and let  $b$ ,  $c$ , and  $d$ , be the other three points in  $M$  which are not on  $l_1$ ,  $l_2$ , or  $l_3$ . It follows from Corollary 3.2 that  $b$ ,  $c$ , and  $d$  must be in one of the planes spanned by two of the long lines containing  $a$ .

Since we are assuming that  $M$  does not contain any 4-point lines, and since any 7-point plane either contains a 4-point line or one of the forbidden minors, then any plane in  $M$  contains at most 6-points. The only way that this could be is that no two of  $b$ ,  $c$ , or  $d$  can be in one of the planes spanned two of the long lines  $l_1$ ,  $l_2$ , or  $l_3$ . Without loss of generality, let  $b$  be in  $l_1 \vee l_2$  plane,  $c$  be in  $l_1 \vee l_3$  plane, and  $d$  be in  $l_2 \vee l_3$  plane.

We now want to show the  $b$ ,  $c$ , and  $d$  are all on three long lines. Let's assume for a contradiction that  $b$  is on two long lines. Then  $M/b$  must be an  $H$ , since it contains seven points and a 3-point line. But one of the points on the 4-point line in an  $H$  is on only one long line. This implies that if  $b$ ,  $c$ , and  $d$  are collinear, then one of the points on  $l_1$  or  $l_2$  must be on only one long line as shown in Figure 5.3a, a contradiction. If  $b$ ,  $c$ , and  $d$  are not collinear, then either  $c$  or  $d$  must be on only one long line as shown in Figure 5.3b, again a contradiction.

Since  $b$ ,  $c$ , and  $d$  are all on three long lines, then the planes  $l_1 \vee l_2$ ,  $l_1 \vee l_3$ ,  $l_2 \vee l_3$ , and  $b \vee c \vee d$  are all  $M(K_4)$ 's, so  $M$  is an  $M(K_5)$ .



Figure 5.3:

We have shown that if  $M$  is a 10-point rank-4 matroid, then  $M$  is an  $O_1$ ,  $O_2$ ,  $R_1$ ,  $R_2$ ,  $R_3$ ,  $P_1(H, H)$ ,  $P_2(H, H)$ ,  $P(H, K_4)$ ,  $M(K_5)$ , or it contains one of the forbidden minors. To finish this case, it suffices to show that any rank-4 one-point extensions of these matroids contains a forbidden minor. By the rank-3 case, any extension of these matroids with an 8-point plane as a minor would contain one of the forbidden minors. By referring back to Figure 2.1 and Figure 2.5, this rules out any possible extensions of an  $O_1$ ,  $O_2$ ,  $R_1$ ,  $R_2$ , or an  $R_3$ .

Now consider the possible rank-4 one-point extensions of an  $M(K_5)$ . By Corollary 3.2, the extra point,  $a$ , would have to be on one of the long lines in the original  $M(K_5)$ . But then  $M/a$  is an 8-point plane, a contradiction.

Now we consider the possible rank-4 one-point extensions of a  $P(H, K_4)$ . If the extended point is not in the  $M(K_4)$  plane, then the extension must contain either a forbidden minor or an  $M(K_5)$ , since if we delete one of the points on the 4-point line, we would have a ten point rank-4 matroid with no 4-point lines. But we just showed



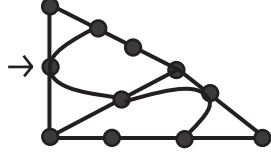


Figure 5.4:

that any rank-4 one-point extension of an  $M(K_5)$  contains a forbidden minor. If the extended point is in the  $M(K_4)$  plane, then that plane must be an  $H$ . But then there exists a point which contracts the matroid to an 8-point plane as shown in Figure 5.4.

Lastly we consider the possible rank-4 one-point extensions of a  $P_1(H, H)$  or a  $P_2(H, H)$ . But any such extension could not be in either  $H$  plane, so we could delete one of the points on the 4-point line, and we would have a ten point rank-4 matroid with no 4-point lines. This would imply that our extension contains an  $M(K_5)$ , but every extension of an  $M(K_5)$  contains a forbidden minor, completing the rank-4 case.

## CHAPTER 6

### Rank-5 Case

Let  $M$  be a rank-5 matroid in  $\mathcal{M}$  and assume that  $|M| = 15$ . Our goal is to show that  $M$  either contains one of the forbidden minors, or that  $M$  is isomorphic to an  $M(K_6)$ .

**Case 1:** Assume that there exists a point  $a$  in  $M$  which is on at least two 4-point lines,  $l_1$  and  $l_2$ , and let  $p_1$  and  $p_2$  be the points in  $M/a$  for which  $a$  contracts  $l_1$  and  $l_2$  to respectively. Since  $M/a$  is connected by the Lemma 3.3, then  $p_1$  and  $p_2$  are contained in a circuit  $C$  which contains at least one other point besides  $p_1$  and  $p_2$ . However,  $C$  can contain only two points that are in the  $l_1 \vee l_2$  plane,  $p_1$  and  $p_2$ , or otherwise the plane  $l_1 \vee l_2$  contains eight points, a contradiction. Therefore, if we contract in  $M$  by all points in  $C$  except for  $p_1$ ,  $p_2$ , and one other point, we get an 8-point plane which is a contradiction. Thus no point in  $M$  can be on two or more 4-point lines.

**Case 2:** Assume for a contradiction that every point in  $M$  is on exactly one 4-point line. Let  $S = \{(a, L) | a \text{ is a point in } M, \text{ and } L \text{ is a 4-point line in } M \text{ which contains } a\}$ . Since there are fifteen points each on exactly one 4-point line, then  $|S| = 15$ . But each line is counted exactly four times, so  $|S|$  is divisible by four, a contradiction. This implies that at least there exists at least one point in  $M$  which is not on any 4-point lines.

**Case 3:** Let  $a$  be a point in  $M$  which is on no 4-point lines. Using Lemma 3.1

and the fact that  $|M/a| \leq 10$ , then  $a$  must be on exactly four 3-point lines,  $l_1, l_2, l_3$ , and  $l_4$ . Since  $|M| = 15$ , we have six points in  $M$  that are not on any of these long lines, and all of these points must be in planes which are spanned by any two of these long lines by Corollary 3.2. Each of the six planes spanned by two of these long lines can contain one or two of these extra points.

Let's assume for a contradiction, that there are two extra points on the plane  $l_1 \vee l_2$ . Since  $l_1$  and  $l_2$  are 3-point lines, then the plane  $l_1 \vee l_2$  must be an  $H$ , so the extra points must both be on a 4-point line. Also, at most one extra point is in the planes  $l_1 \vee l_3$  and  $l_2 \vee l_3$  or else  $M$  would contain an 11-point rank-4 geometry, a contradiction. Similarly, at most one extra point is in the planes  $l_1 \vee l_4$  and  $l_2 \vee l_4$ . But then each of the two extra points in the plane  $l_1 \vee l_2$  is on at most one 4-point line and one 3-point line, and contraction of  $M$  by either point would produce an rank-4 geometry containing at least eleven points, a contradiction.

This means that we must add one point to each of the six planes spanned by two of these long lines, so  $M$  does not contain any 4-point lines. Therefore  $l_1 \vee l_2 \vee l_3$  contains ten points, no 4-point lines, and a point,  $a$ , which is on three 3-point lines, exactly the conditions of case 4 of the rank-4 case, so we can conclude that  $l_1 \vee l_2 \vee l_3$  is an  $M(K_5)$ . Similarly,  $l_1 \vee l_2 \vee l_3$ ,  $l_1 \vee l_3 \vee l_4$ , and  $l_2 \vee l_3 \vee l_4$  must also be  $M(K_5)$ 's which implies that  $M$  is an  $M(K_6)$ .

This means that  $M$  is a rank-5 matroid in  $\mathcal{M}$  containing fifteen points, then  $M$  must be an  $M(K_6)$ . Also, if you added a point to an  $M(K_6)$ , then the new matroid would contain an 11-point rank-4 space. Thus any 16-point rank-5 matroids would have to contain one of the forbidden minors, so the  $M(K_6)$  is the only maximum-sized

rank-5 matroid in  $\mathcal{M}$ .

We are now ready proceed to the inductive step for all ranks greater than 5.

## CHAPTER 7

### Rank- $n$ case

Let  $M$  be a rank- $n$  matroid in  $\mathcal{M}$  such that  $|M| = \binom{n+1}{2}$ , and assume by induction that if  $N$  is a maximum-sized rank- $n - 1$  matroid in  $\mathcal{M}$ , then  $N$  is an  $M(K_n)$ . By using a similar argument as in the first subcase of the rank-5 case, no point in  $M$  can be on two or more 4-point lines. Also, no point can be on more than  $n - 1$  long lines by the Long-Line Lemma. Finally, because  $|M| - |M/a| \geq n$  for every point  $a$  in  $M$ , each point in  $M$  must be on either  $n - 1$  3-point lines; one 4-point line and  $n - 2$  3-point lines; or one 4-point line and  $n - 3$  3-point lines.

**Case 1:** Assume that  $M$  contains a 4-point line. We first show that  $M$  contains a point which is on one 4-point line and  $n - 2$  3-point lines. Suppose there exists a point  $b$  in  $M$  which is on one 4-point line,  $m_1$ , and exactly  $n - 3$  3-point lines,  $m_2, \dots, m_{n-2}$ . Let  $a$  be a point which is not in  $m_1 \vee m_2 \vee \dots \vee m_{n-2}$ . Then  $M/a$  contains a 4-point line and has a rank of at most  $n - 1$ , so by our inductive hypothesis it contains less than  $\binom{n}{2}$  points. Therefore  $a$  is on one 4-point line,  $l_1$ , and  $n - 2$  3-point lines,  $l_2, \dots, l_{n-1}$ .

Let  $N$  be the closed submatroid  $l_1 \vee l_2 \vee \dots \vee l_{n-2}$  of  $M$ . Since  $N$  contains a 4-point line and has a rank of  $n - 1$ , then  $N$  can contain at most  $\binom{n}{2} - 1$  points, by the induction hypothesis. Hence, there are at least  $n - 1$  points in  $M$  not in  $N$  and not on  $l_{n-1}$ . By Corollary 3.2, these points must be in one of the planes  $l_1 \vee l_{n-1}, l_2 \vee l_{n-1}, \dots, l_{n-2} \vee l_{n-1}$ . The plane  $l_1 \vee l_{n-1}$  can contain at most one more

point, so one of the other planes, say  $l_2 \vee l_{n-1}$  must contain two of these extra points. Since  $h(4) = 10$ , each of the planes  $l_3 \vee l_{n-1}, \dots, l_{n-2} \vee l_{n-1}$  can contain at most one extra point. For the same reason, the plane  $l_1 \vee l_{n-1}$  can contain no extra points. Therefore, the number of extra points is at most  $n - 2$ , a contradiction.

**Case 2:** Assume that  $M$  does not contain a 4-point. Then all points in  $M$  must be on exactly  $(n - 1)$  3-point lines. Let's fix a point  $a$  and let  $l_1, \dots, l_{n-1}$  be all the 3-point lines  $a$  is on. Since  $|M| = \binom{n+1}{2}$ , and since the points on  $l_1, \dots, l_{n-1}$  account for  $2n - 1$  of the points of  $M$ , there are  $\binom{n+1}{2} - 2n + 1 = \binom{n-1}{2}$  points in  $M$  not on these long lines. By Corollary 3.2, all the extra points must be in planes spanned by two of these long lines, and since  $M$  contains no 4-point lines, and since there are  $\binom{n-1}{2}$  planes spanned by two of these long lines, each one of these planes must contain six points. This implies that the rank- $(n - 1)$  space spanned by any  $n - 2$  of the long lines containing  $a$ , must contain  $\binom{n+1}{2}$  points and must be an  $M(K_n)$  by our original assumption.

Since  $l_1 \vee l_2 \vee \dots \vee l_{n-2}$  is an  $M(K_n)$ , by Lemma 3.4 we can choose a basis,  $e_1, e_2, \dots, e_{n-1}$ , such that all the points in the  $M(K_n)$  are either basis points, or points of the form  $e_{ij}$  where  $e_{ij}$  is on a 3-point line with  $e_i$  and  $e_j$ , and where  $e_{ij}, e_{jk}$ , and  $e_{ik}$  are collinear for any  $e_i, e_j$ , and  $e_k$  in the basis. Let  $a = e_1$  and then pick the rest of the points in the basis so that  $e_m$  is on  $l_{m-1}$  for all  $2 \leq m \leq n - 1$ . Since  $l_1 \vee l_2 \vee l_{n-1}$  contains ten points and no 4-point lines, then it must be an  $M(K_5)$ . There exists a unique point,  $e_n$ , on  $l_{n-1}$  such that  $e_1, e_2, e_3$ , and  $e_n$  form a basis for this  $M(K_5)$  and satisfy the conditions for this submatroid being an  $M(K_5)$ . By a similar argument, there exists a unique point,  $e_q$ , on  $l_{n-1}$  such that  $e_1, e_3, e_4$ , and  $e_r$  form a basis for

the  $M(K_5)$  on  $l_2 \vee l_3 \vee l_{n-1}$  and satisfies the conditions for this submatroid being an  $M(K_5)$ . But  $e_3$  is on a 3-point line with both  $e_n$  and  $e_r$ , so  $e_n$  and  $e_r$  are the same point. By continuing this process for all possible rank-4 spaces spanned by  $l_{n-1}$  and two other long lines containing  $a$ ,  $l_b$  and  $l_c$ , we get that  $e_n$  is the unique point on  $l_{n-1}$  such that  $e_1, e_{b-1}, e_{c-1}$ , and  $e_n$  form a basis for the  $M(K_5)$  on  $l_b \vee l_c \vee l_{n-1}$  and satisfies the conditions for this submatroid being an  $M(K_5)$ . Therefore,  $e_1, e_2, \dots, e_{n-1}, e_n$  form a basis for  $M$ , and all points in  $M$  are either basis points, or points of the form  $e_{ij}$  where  $e_{ij}$  is on a 3-point line with  $e_i$  and  $e_j$ , and where  $e_{ij}, e_{jk}$ , and  $e_{ik}$  are collinear for any  $e_i, e_j$ , and  $e_k$  in the basis. Thus  $M$  is an  $M(K_{n+1})$  by Lemma 3.4.

We have shown that if  $M$  is a rank- $n$  matroid in  $\mathcal{M}$  which contains  $\binom{n+1}{2}$  points and  $n \geq 5$ , then  $M$  is an  $M(K_{n+1})$ . If you add a point,  $a$  to an  $M(K_{n+1})$ , then it must be on one of the long lines in the  $M(K_{n+1})$  by Corollary 3.2. But then  $M/a > \binom{n}{2}$ , a contradiction. Therefore,  $M(K_{n+1})$  is the only maximum-sized rank- $n$  matroids in  $\mathcal{M}$  for  $n \geq 5$ .

## CHAPTER 8

### Comments and Observations

Using the fact that  $F_7^-$ ,  $P_7$ , and  $H$  all contain the 3-whirl,  $\mathcal{W}_3$ , we obtain the following corollary.

**Corollary 8.1** *Let  $\mathcal{M}$  be the class of all simple matroids which do not contain  $U_{2,5}$ ,  $F_7$ , or  $\mathcal{W}_3$  as a minor. Then  $\mathcal{M}$  has the size function  $h(2) = 4$ ,  $h(3) = 7$ , and  $h(n) = \binom{n+1}{2}$  for  $n \geq 4$ . The maximum-sized matroids in  $\mathcal{M}$  are as follows:*

rank	maximum-sized matroid
2	$U_{2,4}$
3	$P$
4	$O_1, O_2, M(K_5)$
$n \geq 5$	$M(K_{n+1})$

A slightly weaker version of this corollary can be obtained from results of Oxley ([OX2]; see [K2], p.50).



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